arXiv:1004.5213v1 [math-ph] 29 Apr 2010

S-Expansion of Higher-Order Lie Algebras

Ricardo Caroca[1,2],* Nelson Merino[1],† and Patricio Salgado[1‡]

[1] *Departamento de Física, Universidad de Concepción, Casilla 160-C, Concepción, Chile*

[2] *Departamento de Matematica y Física Aplicadas,*

Universidad Católica de la Santisima Concepción,

Alonso de Rivera 2850, Concepción, Chile

(Dated: April 30, 2010)

Abstract

By means of a generalization of the S-expansion method we construct a procedure to obtain expanded higher-order Lie algebras. It is shown that the direct product between an Abelian semigroup S and a higher-order Lie algebra $(\mathcal{G}, [, ...,])$ is also a higher-order Lie algebra. From this S-expanded Lie algebra are obtained resonant submultialgebras and reduced multialgebras of a resonant submultialgebra.

*Electronic address: rcaroca@ucsc.cl

†Electronic address: nemerino@udec.cl

‡Electronic address: pasalgad@udec.cl

Contents

I. INTRODUCTION

Higher-order (or multibracket) simple Lie algebras [1], [2], [3] are generalized ordinary Lie algebras. Their structure constants are given by Lie algebra cohomology cocycles which, by virtue of being such, satisfy a suitable generalization of the Jacobi identity.

As is noted in ref [1], [3] it could be interesting to find applications of these higher-order Lie algebras to know whether the cohomological restrictions which determine and condition their existence have a physical significance. Lie algebra cohomology arguments have already been very useful in various physical problems as in the description of anomalies or in the construction of the Wess-Zumino terms required in the action of extended supersymmetric objects. Other questions may be posed from a purely mathematical point of view. From the discussion in Sect.4 of ref. [1] we know that a representation of a simple Lie algebra may not

be a representation for the associated higher-order Lie algebras. Thus, the representation theory of higher-order algebras requires a separate analysis. A very interesting open problem from a structural point of view is the expansions of higher-order Lie algebras, which will take us outside the domain of the simple ones.

The purpose of this paper is to show that the S-expansion method developed in ref. [4] (see also [5],[6], [7]) can be generalized so that it permits obtaining expanded higher-order Lie algebras.

The paper is organized as follows: In section 2 we shall review some aspects of higher-order Lie algebras. The main point of this section is to display the differences between ordinary Lie algebras and higher-order Lie algebras and to generalize the definitions of higher-order Lie subalgebras and higher-order reduced Lie algebras. In section 3 we generalize the S-expansion method and we show that it is possible to obtain higher-order expanded Lie algebras. In section 4 is shown that, under determined conditions, relevant higher-order Lie subalgebras can be extracted from the S-expanded higher-order Lie algebras.

II. HIGHER-ORDER LIE ALGEBRAS

In this section we shall review some aspects of higher-order Lie algebras. The main point of this section is to display the differences between ordinary Lie algebras and higher-order Lie algebras and to generalize the concepts of subalgebra and reduced Lie algebra of ref. [4].

Definition 1 *An algebra is defined as a pair $(G, \bullet)$ where G is a finite dimensional vector space, and $\bullet : G \times G \to G$ is a rule of composition defined over the vector space.*

Definition 2 *A Lie algebra $\mathcal{G}$ is defined by the pair $(G, [,])$ where G is a finite dimensional vector space , with basis $\{T_A\}_{A=1}^{\dim G}$, over the field K of real or complex numbers; and $[,]$ is a rule of composition $(T_{A_1}, T_{A_2}) \to [T_{A_1}, T_{A_2}] \in G$ which satisfies the following axioms:*

- $[\alpha T_{A_1} + \beta T_{A_2}, T_{A_3}] = \alpha [T_{A_1}, T_{A_3}] + \beta [T_{A_2}, T_{A_3}]$ *for* $\alpha, \beta \in K$ *(linearity),*
- $[T_{A_1}, T_{A_2}] = - [T_{A_2}, T_{A_1}] \qquad \forall\ T_{A_1}, T_{A_2} \in G$ *(antisymmetry),*
- $[[T_{A_1}, T_{A_2}], T_{A_3}] + [[T_{A_2}, T_{A_3}], T_{A_1}] + [[T_{A_3}, T_{A_1}], T_{A_2}] = 0,$

  for all $T_{A_1}, T_{A_2}, T_{A_3} \in G$ *(Jacobi identity).*

The Jacobi identity (JI) can be re-written

$$\frac{1}{1!}\frac{1}{2!}\sum_{\sigma\in S_3}(-1)^{\pi(\sigma)}\left[\left[T_{A_{\sigma(1)}},T_{A_{\sigma(2)}}\right],T_{A_{\sigma(3)}}\right]=0. \tag{1}$$

where S_3 is the permutation group of three elements and $\pi(\sigma)$ is the parity of the permutation σ.

Definition 3 *Let $\mathcal{G}$ be a Lie algebra. A n-bracket $[,...,]$ or skew-symmetric Lie multibracket is a Lie algebra valued n-linear skew-symmetric mapping $[,...,]:\mathcal{G}\times\overset{n}{\ldots}\times\mathcal{G}\to\mathcal{G}$,*

$$(T_{A_1},...,T_{A_n})\to[T_{A_1},...,T_{A_n}]=C^{B}_{A_1...A_n}T_B \tag{2}$$

where the constants $C^{B}_{A_1...A_n}$ are called higher-order structure constants which are completely antisymmetric in the indices $A_1...A_n$.

To define higher-order Lie algebras we need to find the generalization of the Jacobi identity. We postulate that the generalization of the left hand side of eq. (1) is given by

$$\frac{1}{(n-1)!}\frac{1}{n!}\sum_{\sigma\in S_{2n-1}}(-1)^{\pi(\sigma)}\left[\left[T_{A_{\sigma(1)}},...,T_{A_{\sigma(n)}}\right],T_{A_{\sigma(n+1)}},...,T_{A_{\sigma(2n-1)}}\right] \tag{3}$$

However we must find the conditions under which is possible the vanishing of the right hand side. Let T_A be the basis of the algebra in a representation of $\mathcal{G}$. Then is possible to realize the multibracket as

$$\begin{aligned}[T_{A_1},...,T_{A_n}]&=\varepsilon^{B_1...B_n}_{A_1...A_n}T_{B_1}...T_{B_n}\\&=\sum_{\sigma\in S_n}(-1)^{\pi(\sigma)}T_{A_{\sigma(1)}}...T_{A_{\sigma(n)}},\end{aligned} \tag{4}$$

where S_n is the permutation group of n element and $\pi(\sigma)$ is the parity of the permutation σ. In the appendix we will show that the realization (4) of the multibracket satisfy the identity

$$\frac{1}{(n-1)!}\frac{1}{n!}\sum_{\sigma\in S_{2n-1}}(-1)^{\pi(\sigma)}\left[\left[T_{A_{\sigma(1)}},...,T_{A_{\sigma(n)}}\right],T_{A_{\sigma(n+1)}},...,T_{A_{\sigma(2n-1)}}\right] \tag{5}$$

$$=\left\{\begin{array}{ll}0 & ,\ n\text{ even}\\ n\left[T_{A_1},...,T_{A_{2n-1}}\right], & n\text{ odd.}\end{array}\right\}.$$

This means that is possible to obtain a generalization of the Jacobi identity for n even. For n odd we obtain an identity which contains a combination of multibrackets of different orders. Thus we can postulate that [1]

$$\frac{1}{(n-1)!}\frac{1}{n!}\sum_{\sigma\in S_{2n-1}}(-1)^{\pi(\sigma)}\left[\left[T_{A_{\sigma(1)}},...,T_{A_{\sigma(n)}}\right],T_{A_{\sigma(n+1)}},...,T_{A_{\sigma(2n-1)}}\right]=0, \tag{6}$$

is the appropriate generalization of the Jacobi Identity for n even. This identity implies the following condition on the structure constants $C_{A_1...A_n}^{B}$:

$$\varepsilon_{A_1...A_{2n-1}}^{B_1...B_{2n-1}}C_{B_1...B_n}^{C}C_{CB_{n+1}...B_{2n-1}}^{D}=0 \tag{7}$$

which is the generalization of the Jacobi condition [1].

By analogy with the standard Lie algebra, we may now give the following definition [1]:

Definition 4 *Let $\mathcal{G}$ be a Lie algebra and let n be even. A higher-order Lie algebra or multialgebra on $\mathcal{G}$ is the algebra defined by the pair $(\mathcal{G},[,...,])$ where the multibracket $[,...,]$ (2) is multilinear, antisymmetric and satisfies the generalized Jacobi identity (6); and where the higher-order structure constants satisfy the generalized Jacobi condition (7).*

The following definition generalizes the concept of Subalgebra:

Definition 5 ***(Submultialgebra):*** *Let $(\mathcal{G},[,...,])$ be a multialgebra, and consider the Lie algebra $\mathcal{G}$ of the form $\mathcal{G}=V_0\oplus V_1$. The subspace $(V_0,[,...,])$ will be called a submultialgebra of $(\mathcal{G},[,...,])$ if it satisfies*

$$[V_0,V_0,...,V_0]\subset V_0. \tag{8}$$

The existence of submultialgebras is reflected in certain definite restrictions on the structure constants. Let $C_{A_1...A_n}^{B}$ be the generalized structure constants of the multialgebra $(\mathcal{G},[,...,])$. If $\{T_{A_i}\}$, $\left\{T_{a_i^0}\right\}$ and $\left\{T_{a_i^1}\right\}$ denote the bases of $\mathcal{G}$, V_0 and V_1 respectively, where $A_i=1,...,\dim\mathcal{G}$, $a_i^0=1,...,\dim V_0$ and $a_i^1=\dim V_0+1,...,\dim\mathcal{G}$, then the condition (8) can be expressed as

$$C_{a_1^0...a_n^0}^{b^1}=0 \tag{9}$$

for $a_1^0...a_n^0\leq\dim V_0$ and $b^1\geq\dim V_0+1$. In fact, If V_0 is a submultialgebra then $[V_0,V_0,...,V_0]\subset V_0$. This mean that

$$\left[T_{a_1^0},...,T_{a_n^0}\right]=C_{a_1^0...a_n^0}^{b^0}T_{b^0}. \tag{10}$$

i.e. for $\dim V_0 < b^1 < \dim G$ we have $C^{b^1}_{a^0_1 \ldots a^0_n} = 0$.

The following theorem generalizes the concept of reduction of Lie algebras of ref. [4] to higher-order Lie algebras.

Theorem 6 *(Reduced Multialgebra): Let $(\mathcal{G}, [, ...,])$ be a multialgebra, and consider the Lie algebra G of the form $G = V_0 \oplus V_1$, with $\{T_{A_i}\}$ being a basis for G, $\left\{T_{a^0_i}\right\}$ a basis for V_0 and $\left\{T_{a^1_i}\right\}$ a basis for V_1. If the condition*

$$[V_1, V_0, ..., V_0] \subset V_1, \tag{11}$$

is satisfied, then the structure constants $C^{d^0}_{e^1 B_{n+1} \ldots B_{2n-1}}$ are cero, which lead to that the structure constants $C^{b^0}_{a^0_1 \ldots a^0_n}$ satisfy the generalized Jacobi condition by themselves, and therefore

$$\left[T_{a^0_1}, ..., T_{a^0_n}\right] = C^{b^0}_{a^0_1 \ldots a^0_n} T_{b^0} \tag{12}$$

corresponds by itself to a high-order Lie algebra. This algebra, with structure constants $C_{a^0_1 \ldots a^0_n}{}^{b^0}$, is called a reduced multialgebra of $(\mathcal{G}, [, ...,])$ and is symbolized as $|V_0, [, ...,]|$.

Proof. If the condition

$$[V_1, V_0, ..., V_0] \subset V_1$$

is satisfied, we have

$$\begin{aligned}
\left[T_{a^0_1}, ..., T_{a^0_n}\right] &= C^{b^0}_{a^0_1 \ldots a^0_n} T_{b^0} + C^{b^1}_{a^0_1 \ldots a^0_n} T_{b^1} \\
\left[T_{b^1 a^0_1}, ..., T_{a^0_{n-1}}\right] &= C^{c^1}_{b^1 a^0_1 \ldots a^0_{n-1}} T_{c^1} \\
\left[T_{b^1_1}, ..., T_{b^1_n}\right] &= C^{c^0}_{b^1_1 \ldots b^1_n} T_{c^0} + C^{c^1}_{b^1_1 \ldots b^1_n} T_{c^1}
\end{aligned} \tag{13}$$

The structure constant of G satisfy the Jacobi identity

$$\varepsilon^{B_1 \ldots B_{2n-1}}_{A_1 \ldots A_{2n-1}} C^{C}_{B_1 \ldots B_n} C^{D}_{C B_{n+1} \ldots B_{2n-1}} = 0. \tag{14}$$

If $\mathcal{G} = V_0 \oplus V_1$ y $\{T_{A_i}\}$, $\left\{T_{a^0_i}\right\}$, y $\left\{T_{a^1_i}\right\}$ are the corresponding bases of $\mathcal{G}$, V_0, y V_1 (where $A_i = 1, ..., \dim \mathcal{G}$, $a^0_i = 1, ..., \dim V_0$ and $a^1_i = \dim V_0 + 1, ..., \dim \mathcal{G}$), then the generalized Jacobi condition on V_0 is given by

$$\varepsilon^{B_1 \ldots B_{2n-1}}_{a^0_1 \ldots a^0_{2n-1}} C^{E}_{B_1 \ldots B_n} C^{d^0}_{E B_{n+1} \ldots B_{2n-1}} = 0 \tag{15}$$

which can be re-written as

$$\varepsilon^{B_1 \ldots B_{2n-1}}_{a^0_1 \ldots a^0_{2n-1}} C^{e^0}_{B_1 \ldots B_n} C^{d^0}_{e^0 B_{n+1} \ldots B_{2n-1}} + \varepsilon^{B_1 \ldots B_{2n-1}}_{a^0_1 \ldots a^0_{2n-1}} C^{e^1}_{B_1 \ldots B_n} C^{d^0}_{e^1 B_{n+1} \ldots B_{2n-1}} = 0. \tag{16}$$

We consider now the indices $B_1...B_{2n-1}$. If one of these indices takes on a value in V_1, we have

$$\varepsilon^{b_1^1 b_2^0 ... b_{2n-1}^0}_{a_1^0.......a_{2n-1}^0} = \begin{vmatrix} \delta^{b_1^1}_{a_1^0} & \delta^{b_2^0}_{a_1^0} & \cdots & \delta^{b_{2n-1}^0}_{a_1^0} \\ \delta^{b_1^1}_{a_2^0} & \delta^{b_2^0}_{a_2^0} & \cdots & \delta^{b_{2n-1}^0}_{a_2^0} \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{b_1^1}_{a_{2n-1}^0} & \delta^{b_2^0}_{a_{2n-1}^0} & \cdots & \delta^{b_{2n-1}^0}_{a_{2n-1}^0} \end{vmatrix} = \begin{vmatrix} 0 & \delta^{b_2^0}_{a_1^0} & \cdots & \delta^{b_{2n-1}^0}_{a_1^0} \\ 0 & \delta^{b_2^0}_{a_2^0} & \cdots & \delta^{b_{2n-1}^0}_{a_2^0} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \delta^{b_2^0}_{a_{2n-1}^0} & \cdots & \delta^{b_{2n-1}^0}_{a_{2n-1}^0} \end{vmatrix} = 0. \tag{17}$$

From (17) we can see that a column of the determinant is zero and therefore $\varepsilon^{b_1^1 b_2^0 ... b_{2n-1}^0}_{a_1^0.......a_{2n-1}^0} = 0$. Similarly, any permutation on the set $\left(b_1^1 b_2^0 ... b_{2n-1}^0\right)$ in $\varepsilon^{b_1^1 b_2^0 ... b_{2n-1}^0}_{a_1^0.......a_{2n-1}^0}$ will be null. If two indices of the set $(B_1...B_{2n-1})$ take on values in V_1, we have

$$\varepsilon^{b_1^1 b_2^1 b_3^0 ... b_{2n-1}^0}_{a_1^0...........a_{2n-1}^0} = \begin{vmatrix} \delta^{b_1^1}_{a_1^0} & \delta^{b_2^1}_{a_1^0} & \delta^{b_3^0}_{a_1^0} & \cdots & \delta^{b_{2n-1}^0}_{a_1^0} \\ \delta^{b_1^1}_{a_2^0} & \delta^{b_2^1}_{a_2^0} & \delta^{b_3^0}_{a_2^0} & \cdots & \delta^{b_{2n-1}^0}_{a_2^0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \delta^{b_1^1}_{a_{2n-1}^0} & \delta^{b_2^1}_{a_{2n-1}^0} & \delta^{b_3^0}_{a_{2n-1}^0} & \cdots & \delta^{b_{2n-1}^0}_{a_{2n-1}^0} \end{vmatrix} = \begin{vmatrix} 0 & 0 & \delta^{b_3^0}_{a_1^0} & \cdots & \delta^{b_{2n-1}^0}_{a_1^0} \\ 0 & 0 & \delta^{b_3^0}_{a_2^0} & \cdots & \delta^{b_{2n-1}^0}_{a_2^0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \delta^{b_3^0}_{a_{2n-1}^0} & \cdots & \delta^{b_{2n-1}^0}_{a_{2n-1}^0} \end{vmatrix} = 0. \tag{18}$$

From (18) we can see that a column of the determinant is zero and therefore $\varepsilon^{b_1^1 b_2^1 b_3^0 ... b_{2n-1}^0}_{a_1^0...........a_{2n-1}^0} = 0$. In general the number of null columns increase with the number of indices of the set $(B_1...B_{2n-1})$, which take on values in V_1. Thus, the equation (16) is then given by

$$\varepsilon^{b_1^0 ... b_{2n-1}^0}_{a_1^0 ... a_{2n-1}^0} C^{e^0}_{b_1^0 ... b_n^0} C^{d^0}_{e^0 B_{n+1} ... B_{2n-1}} + \varepsilon^{b_1^0 ... b_{2n-1}^0}_{a_1^0 ... a_{2n-1}^0} C^{e^1}_{b_1^0 ... b_n^0} C^{d^0}_{e^1 B_{n+1} ... B_{2n-1}} = 0. \tag{19}$$

From (19) we can see that the structure constant $C^{b^0}_{a_1^0 ... a_n^0}$ satisfy the generalized Jacobi identity by themselves in two cases: ■

- When $C^{e^1}_{b_1^0 ... b_n^0} = 0$, i.e., when V_0 is a submultialgebra
- When $C^{d^0}_{e^1 B_{n+1} ... B_{2n-1}} = 0$, i.e., when $[V_1, V_0, ..., V_0] \subset V_1$. This means that in this case the structure constant $C^{b^0}_{a_1^0 ... a_n^0}$ satisfy the generalized Jacobi identity and

  $$\left[T_{a_1^0}, ..., T_{a_n^0}\right] = C^{b^0}_{a_1^0 ... a_n^0} T_{b^0} \tag{20}$$

  correspond by itself to a higher order Lie algebra. It is interesting to note that a reduced multialgebra $|V_0, [, ...,]|$ does not correspond to a submultialgebra of $(\mathcal{G}, [, ...,])$.

Definition 7 *The Lie multialgebra obtained from the condition* $[V_1, V_0, ..., V_0] \subset V_1$ *i.e., with* $C^{d^0}_{e^1 B_{n+1} ... B_{2n-1}} = 0$ *is called a reduced multialgebra of* G *and will be symbolized as* $\mid V_0 \mid$.

III. S-EXPANSION OF HIGHER-ORDER LIE ALGEBRAS

In this section we shall review some aspects of the S-expansion procedure introduced in ref. [4]. The main point of this section and of this paper is to show that the generalization of the S-expansion method permits obtaining S-expanded higher-order Lie algebras.

A. S-Expansion of Lie Algebras

The S-expansion method is based on combining the structure constants of the Lie algebra $(\mathcal{G},[,])$ with the inner law of a semigroup S to define the Lie bracket of a new, S-expanded algebra. Let $S=\{\lambda_\alpha\}$ be a finite Abelian semigroup endowed with a commutative and associative composition law $S\times S\to S$, $(\lambda_\alpha,\lambda_\beta)\mapsto\lambda_\alpha\lambda_\beta=K_{\alpha\beta}{}^{\gamma}\lambda_\gamma$. **Let** the pair $(\mathcal{G},[,])$ a Lie algebra where G is a finite dimensional vector space, with basis $\{T_A\}_{A=1}^{\dim\mathcal{G}}$, over the field K; and $[,]$ is a ruler of compostion $G\times G\longrightarrow G$, $\left(T_{A_i},T_{A_j}\right)\longrightarrow\left[T_{A_i},T_{A_j}\right]=C_{A_iA_j}^{A_k}T_{A_k}$. The direct product $G=S\otimes G$ is defined as the Cartesian product set

$$\mathfrak{G}=S\times\mathcal{G}=\left\{T_{(A,\alpha)}=\lambda_\alpha T_A:\ \lambda_\alpha\in S\ ,\ T_A\in\mathcal{G}\right\} \tag{21}$$

endowed with a composition law $[,]_S:G\times G\to G$ defined by

$$\left[T_{(A,\alpha)},T_{(B,\beta)}\right]_S=:\lambda_\alpha\lambda_\beta\left[T_A,T_B\right]=K_{\alpha\beta}^{\gamma}C_{AB}^{C}\lambda_\gamma T_C=C_{(A,\alpha)(B,\beta)}^{(C,\gamma)}T_{(C,\gamma)}. \tag{22}$$

where $T_{(A,\gamma)}=\lambda_\gamma T_A$ is a basis of G. The set (21) with the composition law (22) is called a S-expanded Lie algebra. This algebra is a Lie algebra structure defined over the vector space obtained by taking ord S copies of G

$$\mathfrak{G}:\oplus_{\alpha\in S}W_\alpha\ \ (\mathbf{W}_\alpha\approx\mathcal{G},\,\forall\alpha)$$

$\dim G=ordS\times\dim G$ by means of the structure constants

$$C_{(A,\alpha)(B,\beta)}^{(C,\gamma)}=C_{AB}^{C}\delta_{\alpha\beta}^{\gamma} \tag{23}$$

where δ is the Kronecker symbol and the subindex $\alpha,\beta\in S$ denotes the inner compostion in S so that $\delta_{\alpha\beta}^{\gamma}=1$ when $\alpha\beta=\gamma$ in S and zero otherwise. The constants $C_{(A,\alpha)(B,\beta)}^{(C,\gamma)}$ defined by (23) inherit the symmetry properties of C_{AB}^{C} of G by virtue of the abelian character of the S-product, and satisfy the Jacobi identity.

In a nutshell, the S-expansion method can be seen as the natural generalization of the Inönü-Wigner contraction, where instead of to multiply the generators by a numerical parameter, we multiply the generator by the elements of a Abelian semigroup.

Theorem 8 *The product $[,]_S$ defined in (22) is also a Lie product because it is linear, antisymmetric and satisfies the Jacobi identity. This product defines a new Lie algebra characterized by the pair $(\mathfrak{G},[,]_S)$, and is called a S-expanded Lie algebra.*

Proof. Since the S-product is abelian, the product $[,]_S$ defined by (22) inherits the symmetry properties of the product $[,]$ of $\mathcal{G}$, and satisfies the Jacobi identity. In fact,

$$\begin{aligned}
&\left[\left[T_{(A_1,\alpha_1)},T_{(A_2,\alpha_2)}\right]_S,T_{(A_3,\alpha_3)}\right]_S+\left[\left[T_{(A_2,\alpha_2)},T_{(A_3,\alpha_3)}\right]_S,T_{(A_1,\alpha_1)}\right]_S\\
&+\left[\left[T_{(A_3,\alpha_3)},T_{(A_1,\alpha_1)}\right]_S,T_{(A_2,\alpha_2)}\right]_S\\
&=\frac{1}{1!}\frac{1}{2!}\sum_{\sigma\in S_3}(-1)^{\pi(\sigma)}\left[\left[T_{\left(A_{\sigma(1)},\alpha_{\sigma(1)}\right)},T_{\left(A_{\sigma(2)},\alpha_{\sigma(2)}\right)}\right]_S,T_{\left(A_{\sigma(3)},\alpha_{\sigma(3)}\right)}\right]_S\\
&=\frac{1}{1!}\frac{1}{2!}\sum_{\sigma\in S_3}(-1)^{\pi(\sigma)}\lambda_{\alpha_{\sigma(1)}}\lambda_{\alpha_{\sigma(2)}}\lambda_{\alpha_{\sigma(3)}}\left[\left[T_{A_{\sigma(1)}},T_{A_{\sigma(2)}}\right],T_{A_{\sigma(3)}}\right]\\
&=\frac{1}{1!}\frac{1}{2!}\sum_{\sigma\in S_3}(-1)^{\pi(\sigma)}K_{\alpha_{\sigma(1)}\alpha_{\sigma(2)}\alpha_{\sigma(3)}}^{\gamma}\lambda_\gamma\left[\left[T_{A_{\sigma(1)}},T_{A_{\sigma(2)}}\right],T_{A_{\sigma(3)}}\right]\\
&=K_{\alpha_1\alpha_2\alpha_3}^{\gamma}\lambda_\gamma\left(\frac{1}{1!}\frac{1}{2!}\sum_{\sigma\in S_3}(-1)^{\pi(\sigma)}\left[\left[T_{A_{\sigma(1)}},T_{A_{\sigma(2)}}\right],T_{A_{\sigma(3)}}\right]\right)=0
\end{aligned}\tag{24}$$

where we have used the commutativity $\left(K_{\alpha_{\sigma(1)}\alpha_{\sigma(2)}\alpha_{\sigma(3)}}^{\gamma}=K_{\alpha_1\alpha_2\alpha_3}^{\gamma}\right)$ and associativity of the semigroup inner law, and the fact that the product $[,]$ satisfies the Jacobi identity. ■

From (24) we can see that the Jacobi identity of the S-expanded Lie algebra $(S\otimes\mathcal{G},[,]_S)$

$$\left(\begin{array}{c}\left[\left[T_{(A_1,\alpha_1)},T_{(A_2,\alpha_2)}\right]_S,T_{(A_3,\alpha_3)}\right]_S+\left[\left[T_{(A_2,\alpha_2)},T_{(A_3,\alpha_3)}\right]_S,T_{(A_1,\alpha_1)}\right]_S\\ +\left[\left[T_{(A_3,\alpha_3)},T_{(A_1,\alpha_1)}\right]_S,T_{(A_2,\alpha_2)}\right]_S\end{array}\right)=0\tag{25}$$

can be obtained if we multiply the Jacobi identity of the Lie algebra $(\mathcal{G},[,])$ by $\lambda_{\alpha_1}\lambda_{\alpha_2}\lambda_{\alpha_3}$ or by the 3-selector $K_{\alpha_1\alpha_2\alpha_3}^{\gamma}$:

$$\mathrm{JI}\left(S\otimes\mathcal{G},[,]_S\right)=K_{\alpha_1\alpha_2\alpha_3}^{\gamma}\left(\mathrm{JI}\left(\mathcal{G},[,]\right)\right)\ .\tag{26}$$

Similarly, if multiply the Jacobi condition of the Lie algebra $(\mathcal{G},[,])$

$$\frac{1}{2}\varepsilon_{A_1A_2A_3}^{B_1B_2B_3}C_{B_1B_2}^{C}C_{CB_3}^{D}=0\tag{27}$$

by $K_{\alpha_1\alpha_2\alpha_3}^{\beta} = K_{\alpha_1\alpha_2}^{\gamma} K_{\gamma\alpha_3}^{\beta}$, we obtain the Jacobi condition of the S-expanded Lie algebra $(S \otimes \mathcal{G}, [,]_S)$. In fact,

$$K_{\alpha_1\alpha_2\alpha_3}^{\beta} \left(\frac{1}{2} \varepsilon_{A_1A_2A_3}^{B_1B_2B_3} C_{B_1B_2}^{C} C_{CB_3}^{D} \right) = \frac{1}{2} \varepsilon_{A_1A_2A_3}^{B_1B_2B_3} K_{\alpha_1\alpha_2}^{\gamma} C_{B_1B_2}^{C} K_{\gamma\alpha_3}^{\beta} C_{CB_3}^{D} = 0 \tag{28}$$

$$\frac{1}{2} \varepsilon_{A_1A_2A_3}^{B_1B_2B_3} C_{(B_1,\alpha_1)(B_2,\alpha_2)}^{(C,\gamma)} C_{(C,\gamma)(B_3,\alpha_3)}^{(D,\beta)} = 0. \tag{29}$$

B. S-Expansion of Lie Multialgebras

The S-expansion method is based on combining the structure constants of $(\mathcal{G}, [, ...,])$ with the inner law of a semigroup S to define the Lie bracket of a new, S-expanded multialgebra. Let $S = \{\lambda_\alpha\}$ be a finite Abelian semigroup endowed with a commutative and associative composition law $S \times S \to S$, $(\lambda_\alpha, \lambda_\beta) \mapsto \lambda_\alpha \lambda_\beta = K_{\alpha\beta}{}^{\gamma} \lambda_\gamma$. The direct product $G = S \otimes G$ is defined as the cartesian product set

$$\mathfrak{G} = S \times \mathcal{G} = \left\{ T_{(A,\alpha)} = \lambda_\alpha T_A : \lambda_\alpha \in S \ , T_A \in \mathcal{G} \right\} \tag{30}$$

with the composition law $[, ...,]_S : G \overset{n}{\times ... \times} G \to G$, defined by

$$\left[T_{(A_1,\alpha_1)}, ..., T_{(A_n,\alpha_n)} \right]_S = \lambda_{\alpha_1} ... \lambda_{\alpha_n} \left[T_{A_1}, ..., T_{A_n} \right]$$

$$\left[T_{(A_1,\alpha_1)}, ..., T_{(A_n,\alpha_n)} \right]_S = K_{\alpha_1...\alpha_n}^{\gamma} C_{A_1...A_n}^{C} \lambda_\gamma T_C = C_{(A_1,\alpha_1)...(A_n,\alpha_n)}^{(C,\gamma)} T_{(C,\gamma)} \tag{31}$$

where $T_{(A_i,\alpha_i)} \in G$, $\forall i = 1, ..., n$, and $C_{(A_1,\alpha_1)...(A_n,\alpha_n)}^{(C,\gamma)} = K_{\alpha_1...\alpha_n}^{\gamma} C_{A_1...A_n}^{C}$.

The set $G = S \times G$ (30) with the composition law (31) define a new Lie multialgebra which will be called S-expanded Lie multialgebra. This algebra is a Lie algebra structure defined over the vector space obtained by taking S copies of G by means of the structure constant $C_{(A_1,\alpha_1)...(A_n,\alpha_n)}^{(C,\gamma)} = K_{\alpha_1...\alpha_n}^{\gamma} C_{A_1...A_n}^{C}$ where $K_{\alpha_1...\alpha_n}^{\gamma} = K_{\alpha_1...\alpha_{n-1}}^{\sigma} K_{\sigma\alpha_n}^{\gamma}$. The structure constants $C_{(A_1,\alpha_1)...(A_n,\alpha_n)}^{(C,\gamma)}$ defined in (31) inherit the symmetry properties of $C_{A_1...A_n}^{C}$ of G by virtue of the abelian character of the S-product.

Theorem 9 *The product $[, ...,]_S$ defined in (31)) is multilinear, antisymmetric and satisfies the generalized Jacobi identity (GJI).*

$$a \sum_{\sigma \in S_{2n-1}} (-1)^{\pi(\sigma)} \left[\left[T_{\left(A_{\sigma(1)},\alpha_{\sigma(1)}\right)}, .., T_{\left(A_{\sigma(n)},\alpha_{\sigma(n)}\right)} \right]_S, T_{\left(A_{\sigma(n+1)},\alpha_{\sigma(n+1)}\right)}, .., T_{\left(A_{\sigma(2n-1)},\alpha_{\sigma(2n-1)}\right)} \right]_S = 0 \tag{32}$$

where

$$a=\frac{1}{(n-1)!}\frac{1}{n!}$$

Proof. Since the S-product is abelian, the product $[,...,]_S$ defined by (31) inherits the symmetry properties of the product $[,...,]$ of $(\mathcal{G},[,...,])$, and satisfies the generalized Jacobi identity. In fact,

$$\begin{aligned}
&\sum_{\sigma\in S_{2n-1}}(-1)^{\pi(\sigma)}\left[\left[T_{\left(A_{\sigma(1)},\alpha_{\sigma(1)}\right)},...,T_{\left(A_{\sigma(n)},\alpha_{\sigma(n)}\right)}\right]_S,T_{\left(A_{\sigma(n+1)},\alpha_{\sigma(n+1)}\right)},...,T_{\left(A_{\sigma(2n-1)},\alpha_{\sigma(2n-1)}\right)}\right]_S\\
&=\sum_{\sigma\in S_{2n-1}}(-1)^{\pi(\sigma)}\lambda_{\alpha_{\sigma(1)}}\ldots\lambda_{\alpha_{\sigma(2n-1)}}\left[\left[T_{A_{\sigma(1)}},...,T_{A_{\sigma(n)}}\right],T_{A_{\sigma(n+1)}},...,T_{A_{\sigma(2n-1)}}\right]\\
&=\sum_{\sigma\in S_{2n-1}}(-1)^{\pi(\sigma)}K^{\gamma}_{\alpha_{\sigma(1)}\cdots\alpha_{\sigma(2n-1)}}\lambda_\gamma\left[\left[T_{A_{\sigma(1)}},...,T_{A_{\sigma(n)}}\right],T_{A_{\sigma(n+1)}},...,T_{A_{\sigma(2n-1)}}\right]\\
&=K^{\gamma}_{\alpha_1\ldots\alpha_{2n-1}}\lambda_\gamma\left(\sum_{\sigma\in S_{2n-1}}(-1)^{\pi(\sigma)}\left[\left[T_{A_{\sigma(1)}},...,T_{A_{\sigma(n)}}\right],T_{A_{\sigma(n+1)}},...,T_{A_{\sigma(2n-1)}}\right]\right)=0,
\end{aligned}\tag{33}$$

where we have used the commutativity $K^{\gamma}_{\alpha_{\sigma(1)}\cdots\alpha_{\sigma(2n-1)}}=K^{\gamma}_{\alpha_1\ldots\alpha_{2n-1}}$ and associativity of the semigroup inner law, and the fact that the product $[,...,]$ satisfies the generalized Jacobi identity. ■

From (33) we can see that the Jacobi identity of the S-expanded Lie multialgebra $(S\otimes\mathcal{G},[,...,]_S)$ can be obtained if we multiply the generalized Jacobi identity of the Lie multialgebra $(\mathcal{G},[,...,])$ by $K^{\gamma}_{\alpha_1\ldots\alpha_{2n-1}}$.

Similarly, if we multiply the generalized Jacobi condition of the Lie algebra $(\mathcal{G},[,...,])$

$$\varepsilon^{B_1\ldots B_{2n-1}}_{A_1\ldots A_{2n-1}}C^{C}_{B_1\ldots B_n}C^{D}_{CB_{n+1}\ldots B_{2n-1}}=0\tag{34}$$

by $K^{\beta}_{\alpha_1\ldots\alpha_{2n-1}}=K^{\gamma}_{\alpha_1\ldots\alpha_n}K^{\beta}_{\gamma\alpha_{n+1}\ldots\alpha_{2n-1}}$, we obtain the generalized Jacobi condition of the S-expanded Lie multialgebra $(\mathfrak{G},[,...,]_S)$. In fact,

$$K^{\beta}_{\alpha_1\ldots\alpha_{2n-1}}\left(\varepsilon^{B_1\ldots B_{2n-1}}_{A_1\ldots A_{2n-1}}C^{C}_{B_1\ldots B_n}C^{D}_{CB_{n+1}\ldots B_{2n-1}}\right)=0\tag{35}$$

$$\varepsilon^{B_1\ldots B_{2n-1}}_{A_1\ldots A_{2n-1}}K^{\gamma}_{\alpha_1\ldots\alpha_n}C^{C}_{B_1\ldots B_n}K^{\beta}_{\gamma\alpha_{n+1}\ldots\alpha_{2n-1}}C^{D}_{CB_{n+1}\ldots B_{2n-1}}=0$$

$$\varepsilon^{B_1\ldots B_{2n-1}}_{A_1\ldots A_{2n-1}}C^{(C,\gamma)}_{(B_1,\alpha_1)\ldots(B_n,\alpha_n)}C^{D}_{(C,\gamma)(B_{n+1},\alpha_{n+1})\ldots(B_{2n-1},\alpha_{n+1})}=0.\tag{36}$$

C. Multialgebra 0_S-Reduced

When the semigroup has a zero element $0_S\in S$, it plays a somewhat peculiar role in the S-expanded Lie multialgebra. Let us span S in nonzero elements $\lambda_i, i=0,\cdots,N$, and a zero

element $\lambda_{N+1} = 0_S$, i.e.,

$$S = \left\{ \underbrace{\lambda_0, \lambda_1, \ldots, \lambda_N}_{\lambda_i}, \underbrace{\lambda_{N+1}}_{0_S} \right\}. \tag{37}$$

Then, the 2-selector satisfies

$$K_{N+1,i_2,...,i_n}{}^{j} = K_{\underbrace{N+1,...,N+1}_{r},i_{r+1},...,i_n}{}^{j} = K_{\underbrace{N+1,...,N+1}_{r},i_{r+1},...,i_n}{}^{N+1} = \cdots = K_{N+1,...,N+1}{}^{j} = 0$$

$$K_{N+1,i_2,...,i_n}{}^{N+1} = K_{N+1,...,N+1}{}^{N+1} = 1. \tag{38}$$

Therefore, the S-expanded multialgebra $(\mathfrak{G}, [,...,]_S)$ can be split as

$$\begin{aligned}
\left[T_{(A_1,i_1)}, \ldots, T_{(A_n,i_n)}\right]_S &= K_{i_1,...,i_n}{}^{k} C_{A_1 \ldots A_n}{}^{C} T_{(C,k)} + K_{i_1,...,i_n}{}^{N+1} C_{A_1,...,A_n}{}^{C} T_{(C,N+1)} \\
\left[T_{(A_1,N+1)}, T_{(A_2,i_2)}, \ldots, T_{(A_n,i_n)}\right]_S &= C_{A_1,...,A_n}{}^{C} T_{(C,N+1)} \\
&\vdots \\
\left[T_{(A_1,N+1)}, \ldots, T_{(A_r,N+1)}, T_{(A_{r+1},i_{r+1})}, \ldots, T_{(A_n,i_n)}\right]_S &= C_{A_1,...,A_n}{}^{C} T_{(C,N+1)} \\
&\vdots \\
\left[T_{(A_1,N+1)}, \ldots, T_{(A_n,N+1)}\right]_S &= C_{A_1,...,A_n}{}^{C} T_{(C,N+1)}.
\end{aligned} \tag{39}$$

From (39) we can see that $(\mathfrak{G}, [,...,]_S)$ can be written as $\mathfrak{G} = V_0 \oplus V_1$, with $V_0 = \{T_{(A,i)}\}$, $V_1 = \{T_{(A,N+1)}\}$. From (39) we also see that

$$[V_1, V_0, ..., V_0]_S \subset V_1 \tag{40}$$

$$\Big[\underbrace{V_1, ..., V_1}_{r\text{-times}}, V_0, ..., V_0\Big]_S \subset V_1, \quad \text{con } r = 1, ..., n. \tag{41}$$

This means that the commutation relations

$$\left[T_{(A_1,i_1)}, \ldots, T_{(A_n,i_n)}\right]_S = K_{i_1,...,i_n}{}^{k} C_{A_1 \ldots A_n}{}^{C} T_{(C,k)}$$

are those of a reduced Lie multialgebra $(\mathfrak{G}, [,...,]_S)$. From (39) we see that the reduction procedure in this particular case is equivalent to imposing the condition

$$T_{(C,N+1)} = 0_S T_C = 0.$$

The above considerations motivate the following definition:

Definition 10 *Let S be an Abelian semigroup with a zero element $0_S \in S$, and let $(S \otimes \mathcal{G}, [,...,])$ be an S-expanded multialgebra. The multialgebra obtained by imposing the condition $0_S T_A = 0$ on $\mathfrak{G}$ is called a 0_S-reduced multialgebra of $\mathfrak{G}$.*

IV. S-EXPANSION OF SUBMULTIALGEBRAS

In this section is shown that there are at least two ways of extracting smaller multialgebras from $(S \otimes \mathcal{G}, [, ...,])$. The first one gives rise to a "resonant submultialgebra" while the second produces reduced multialgebras of a resonant submultialgebra.

A. Resonant submultialgebras

The general problem of finding submultialgebras from an S-expanded multialgebra is a nontrivial one, which is met and solved in this section. In order to provide a solution, one must have some information about the subspace structure of $\mathcal{G}, [, ...,]$. This information is encoded in the following way:

Let $\mathcal{G} = \oplus_{p\in I} V_p$ be a decomposition of $\mathcal{G}$ in subspaces V_p, where I is a set of indices. For each $(p_1, ..., p_n) \in I$ it is always possible to define $i_{(p_1,...,p_n)} \subset I$ such that

$$[V_{p_1}, ..., V_{p_n}] \subset \bigoplus_{r\in i_{(p_1,...,p_n)}} V_r. \tag{42}$$

In this way, the subsets $\left\{ i_{(p_1,...,p_n)} \right\}$ store the information on the subspace structure of $\mathcal{G}$.

As for the Abelian semigroup S, this can always be decomposed as $S = \cup_{p\in I} S_p$, where $S_p \subset S$. In principle, this decomposition is completely arbitrary; however, using the product from definition (2.2) of ref. [4], it is sometimes possible to pick out a very particular choice of subset decomposition. This choice is the subject of the following definition:

Definition 11 *Let $\mathcal{G} = \oplus_{p\in I} V_p$ be a decomposition of $\mathcal{G}$ in subspaces V_p, with a structure described by the subsets $i_{(p_1,...,p_n)}$, as in Eq.(42). Let $S = \cup_{p\in I} S_p$ be a subset decomposition of the Abelian semigroup S such that*

$$S_{p_1} \times S_{p_2} \times \cdots \times S_{p_n} \subset \bigcap_{r\in i_{(p_1,...,p_n)}} S_r. \tag{43}$$

When such a subset decomposition $S = \cup_{p\in I} S_p$ exists, then we say that this decomposition is in resonance with the subspace decomposition of $\mathcal{G} = \oplus_{p\in I} V_p$.

Theorem 12 *Let $\mathcal{G} = \oplus_{p\in I} V_p$ be a subspace decomposition of $\mathcal{G}$,with a structure described by Eq. (42), and let $S = \cup_{p\in I} S_p$ be a resonant subset decomposition of the Abelian semigroup*

S, with the structure given in Eq.(43). Define the subspaces W_p of $\mathfrak{G} = S \otimes \mathcal{G}$,

$$W_p = S_p \otimes V_p,\ \ p \in I. \tag{44}$$

Then,

$$\mathfrak{G}_R = \oplus_{p \in I} W_p \tag{45}$$

is called a resonant subalgebra of the S-expanded multialgebra $\mathfrak{G} = S \otimes \mathcal{G}$.

Proof. Using Eqs. (42) and (43) we have

$$\begin{aligned} [W_{p_1}, ..., W_{p_n}]_S &= [S_{p_1} \otimes V_{p_1}, ..., S_{p_n} \otimes V_{p_n}]_S = (S_{p_1} \times ... \times S_{p_n}) \otimes [V_{p_1}, ..., V_{p_n}] \\ &\subset \left(\bigcap_{s \in i_{(p_1,...,p_n)}} S_s \right) \otimes \left(\bigoplus_{r \in i_{(p_1,...,p_n)}} V_r \right) = \bigoplus_{r \in i_{(p_1,...,p_n)}} \left(\bigcap_{s \in i_{(p_1,...,p_n)}} S_s \right) \otimes V_r. \end{aligned} \tag{46}$$

But, it is clear that for each $r \in i_{(p_1,...,p_n)}$ one can write

$$\bigcap_{s \in i_{(p_1,...,p_n)}} S_s \subset S_r. \tag{47}$$

Then,

$$\begin{aligned} [W_{p_1}, ..., W_{p_n}]_S &\subset \bigoplus_{r \in i_{(p_1,...,p_n)}} S_r \otimes V_r = \bigoplus_{r \in i_{(p_1,...,p_n)}} W_r \\ [W_{p_1}, ..., W_{p_n}]_S &\subset \bigoplus_{r \in i_{(p_1,...,p_n)}} S_r \otimes V_r = \bigoplus_{r \in i_{(p_1,...,p_n)}} W_r \\ [W_{p_1}, ..., W_{p_n}]_S &\subset \bigoplus_{r \in I} W_r = \mathfrak{G}_R \end{aligned} \tag{48}$$

■

Therefore, the algebra closes and $\mathfrak{G}_R$ is a submultialgebra of $\mathfrak{G}$.

This theorem translates the difficult problem of finding subalgebras from an S-expanded algebra $\mathfrak{G} = S \otimes \mathfrak{g}$ into that of finding a resonant partition for the semigroup S.

Denoting the basis of V_{p_i} by $\{T_{a_{p_i}}\}$, $\lambda_{\alpha_{p_i}} \in S_{p_i}$ and $T_{\left(a_{p_i}, \alpha_{p_i}\right)} = \lambda_{\alpha_{p_i}} T_{a_{p_i}} \in W_{p_i}$ one can write

$$\left[T_{\left(a_{p_1}, \alpha_{p_1}\right)}, ..., T_{\left(a_{p_n}, \alpha_{p_n}\right)} \right]_S = C_{\left(a_{p_1}, \alpha_{p_1}\right) ... \left(a_{p_n}, \alpha_{p_n}\right)}{}^{(c_r, \gamma_r)} T_{(c_r, \gamma_r)},$$

which means that the structure constants of the resonant submultialgebra are given by

$$C_{\left(a_{p_1}, \alpha_{p_1}\right) ... \left(a_{p_n}, \alpha_{p_n}\right)}{}^{(c_r, \gamma_r)} = K_{\alpha_{p_1} ... \alpha_{p_n}}{}^{\gamma_r} C_{a_{p_1} ... a_{p_n}}{}^{c_r}.$$

An interesting fact is that the S-expanded multialgebra "subspace structure" encoded in $i_{(p_1,...,p_n)}$ is the same as in the original multialgebra, as can be observed fron Eq. (48).

B. Reduced Multialgebras of a Resonant Submultialgebra

The following theorem provides necessary conditions under which a reduced multialgebra can be extracted from a resonant subalgebra:

Theorem 13 *Let $\mathfrak{G}_R = \oplus_{p\in I} S_p \otimes V_p$ be a resonant submultialgebra $(\mathfrak{G}, [,...,]_S)$, i.e., let Eqs. (42) and (43) be satisfied. Let $S_p = \hat{S}_p \cup \check{S}_p$ be a partition of the subsets $S_p \subset S$ such that*

$$\check{S}_{p_i} \cap \hat{S}_{p_i} = \phi \tag{49}$$

$$\hat{S}_{p_1} \times \check{S}_{p_2} \times ... \times \check{S}_{p_n} \subset \bigcap_{r\in i_{(p_1,...,p_n)}} \hat{S}_r. \tag{50}$$

The conditions (49) and (50) induce the decomposition $\mathfrak{G}_R = \check{\mathfrak{G}}_R \oplus \overset{\wedge}{\mathfrak{G}}_R$ on the resonant subalgebra, where

$$\check{\mathfrak{G}}_R = \oplus_{p\in I} \check{S}_p \otimes V_p \tag{51}$$

$$\overset{\wedge}{\mathfrak{G}}_R = \oplus_{p\in I} \hat{S}_p \otimes V_p. \tag{52}$$

When conditions (49) and (50) hold, then

$$\left[\overset{\wedge}{\mathfrak{G}}_R, \check{\mathfrak{G}}_R, ..., \check{\mathfrak{G}}_R\right]_S \subset \overset{\wedge}{\mathfrak{G}}_R \tag{53}$$

and therefore $\left|\check{\mathfrak{G}}_R\right|$ corresponds to a reduced algebra of $\mathfrak{G}_R$.

Proof. $\hat{W}_{p_i} = \hat{S}_{p_i} \otimes V_{p_i}$ and $\check{W}_{p_i} = \check{S}_{p_i} \otimes V_{p_i}$. Then, using condition (50), we have:

$$\begin{aligned}
\left[\hat{W}_{p_1}, \check{W}_{p_2}, ..., \check{W}_{p_n}\right]_S &= \left[\hat{S}_{p_1} \otimes V_{p_1}, \check{S}_{p_2} \otimes V_{p_2}, ..., \check{S}_{p_n} \otimes V_{p_n}\right]_S \\
&= \left(\hat{S}_{p_1} \times \check{S}_{p_2} \times ... \times \check{S}_{p_n}\right) \otimes [V_{p_1}, V_{p_2}, ..., V_{p_n}] \\
&\subset \left(\bigcap_{s\in i_{(p_1,...,p_n)}} \hat{S}_s\right) \otimes \left(\bigoplus_{r\in i_{(p_1,...,p_n)}} V_r\right) \\
&= \bigoplus_{r\in i_{(p_1,...,p_n)}} \left(\bigcap_{s\in i_{(p_1,...,p_n)}} \hat{S}_s\right) \otimes V_r.
\end{aligned}$$

For each $r \in i_{(p_1,...,p_n)}$ we have

$$\bigcap_{s\in i_{(p_1,...,p_n)}} \hat{S}_s \subset \hat{S}_r$$

so that,

$$\left[\hat{W}_{p_1}, \check{W}_{p_2}, ..., \check{W}_{p_n}\right]_S \subset \bigoplus_{r\in i_{(p_1,...,p_n)}} \hat{S}_r \otimes V_r = \bigoplus_{r\in i_{(p_1,...,p_n)}} \hat{W}_r$$
$$\subset \bigoplus_{r\in I} \hat{W}_r = \overset{\wedge}{\mathfrak{G}}_R.$$

Thus $\left[\hat{W}_{p_1}, \check{W}_{p_2}, ..., \check{W}_{p_n}\right]_S \subset \overset{\wedge}{\mathfrak{G}}_R$, i.e,

$$\left[\overset{\wedge}{\mathfrak{G}}_R, \check{\mathfrak{G}}_R, ..., \check{\mathfrak{G}}_R\right]_S \subset \overset{\wedge}{\mathfrak{G}}_R$$

and therefore $\left|\check{\mathfrak{G}}_R\right|$ is a reduced algebra of $\mathfrak{G}_R$. ■

The structure constants for the reduced algebra $\left|\check{\mathfrak{G}}_R\right|$ are given by,

$$C_{\left(a_{p_1},\alpha_{p_1}\right)...\left(a_{p_n},\alpha_{p_n}\right)}{}^{(c_r,\gamma_r)} = K_{\alpha_{p_1}...\alpha_{p_n}}{}^{\gamma_r} C_{a_{p_1}...a_{p_n}}{}^{c_r}$$

with α_{p_i}, γ_r such that $\lambda_{\alpha_{p_i}} \in \check{S}_{p_i}$ y $\lambda_{\gamma_r} \in \check{S}_{p_r}$.

C. $S_E^{(N)}$-Expansion of Multialgebras

Definition 14 *Let us define $S_E^{(N)}$ as the semigroup of elements [8]*

$$S_E^{(N)} = \{\lambda_\alpha,\ \alpha = 0, ..., N+1\} \tag{54}$$

provided with a multiplication rule

$$\lambda_\alpha \lambda_\beta = \lambda_{H_{N+1}(\alpha+\beta)} = \delta^{\gamma}_{H_{N+1}(\alpha+\beta)} \lambda_\gamma \tag{55}$$

where H_{N+1} is defined as the function

$$H_n(x) = \left\{ \begin{array}{l} x,\ when\ x < n, \\ n,\ when\ x \geq n. \end{array} \right\}. \tag{56}$$

The two-selectors for $S_E^{(N)}$ read

$$K^{\gamma}_{\alpha\beta} = \delta^{\gamma}_{H_{N+1}(\alpha+\beta)}$$

where δ^{ρ}_{σ} is the Kronecker delta.

The multiplication rule (55) can be directly generalized to

$$\lambda_{\alpha_1}....\lambda_{\alpha_n} = \lambda_{H_{N+1}(\alpha_1+...+\alpha_n)} = \delta^{\gamma}_{H_{N+1}(\alpha_1+...+\alpha_n)}\lambda_\gamma \tag{57}$$
$$K_{\alpha_1...\alpha_n}{}^{\gamma} = \delta^{\gamma}_{H_{N+1}(\alpha_1+...+\alpha_n)}.$$

From Eq.(55), we have that λ_{N+1} is the zero element in $S_E^{(N)}$, i.e., $\lambda_{N+1} = 0_S$.

The corresponding S-expanded multialgebra is given by the following commutation relation:

$$\left[T_{(A_1,\alpha_1)}, ..., T_{(A_n,\alpha_n)}\right]_S = \delta^{\gamma}_{H_{N+1}(\alpha_1+...+\alpha_n)} C_{A_1...A_n}{}^{C} T_{(C,\gamma)}, \tag{58}$$

which implies that the structure constants for the $S_E^{(N)}$-expanded multialgebra can be written as

$$C_{(A_1,\alpha_1)...(A_n,\alpha_n)}{}^{(C,\gamma)} = \delta^{\gamma}_{H_{N+1}(\alpha_1+...+\alpha_n)} C_{A_1...A_n}{}^{C} \tag{59}$$

with $\gamma, \alpha_1, ..., \alpha_n = 0, \cdots, N+1$. When the condition of 0_S-reduction is imposed, the Eq. (59) reduces to

$$C_{(A_1,i_1)...(A_n,i_n)}{}^{(C,k)} = \delta^{k}_{H_{N+1}(i_1+...+i_n)} C_{A_1...A_n}{}^{C}.$$

V. COMMENTS

We have shown that the successful S-expansion of the Lie algebras method, developed in ref. [4], can be generalized so as to obtain expanded higher-order Lie algebras.

The main results of this paper are: the generalizations of the definitions of Lie subalgebras and reduced Lie algebras to higher-order Lie subalgebras and higher-order reduced Lie algebras; to generalize the S-expansion method and to show that it is possible to obtain higher-order expanded Lie algebras, as well as to probe that under determined conditions can be extracted relevant higher-order Lie subalgebras from the S-expanded higher-order Lie algebras.

This work was supported in part by FONDECYT through Grants #s 1080530 and 1070306 and in part by Dirección de Investigación, Universidad de Concepción through Grant # 208.011.048-1.0. One of the authors (P.S) wish to thank J.A. de Azcarraga for his kind hospitality at the Departament of Theoretical Physics of Valencia University and many enlightening discussions.Two of the authors (R.C. and N.M) were supported by grants from the Comisión Nacional de Investigación Científica y Tecnológica CONICYT and from the Universidad de Concepción, Chile.

VI. APPENDIX A

In this appendix we show that the realization (4) of the multibracket satisfies the identity

$$\frac{1}{(n-1)!}\frac{1}{n!}\sum_{\sigma\in S_{2n-1}}(-1)^{\pi(\sigma)}\left[\left[T_{A_{\sigma(1)}},...,T_{A_{\sigma(n)}}\right],T_{A_{\sigma(n+1)}},...,T_{A_{\sigma(2n-1)}}\right] \tag{60}$$

$$=\left\{\begin{array}{ll} 0 & ,\ n \text{ even} \\ n\left[T_{A_1},...,T_{A_{2n-1}}\right], & n \text{ odd.}\end{array}\right\}$$

which can be re-written in the following way:

$$\frac{1}{(n-1)!}\frac{1}{n!}\varepsilon^{B_1...B_{2n-1}}_{A_1...A_{2n-1}}\left[\left[T_{B_1},...,T_{B_n}\right],T_{B_{n+1}},...,T_{B_{2n-1}}\right] \tag{61}$$

$$=\left\{\begin{array}{ll} 0 & ,\ n \text{ even} \\ nn!\,(n-1)!\left[T_{A_1},...,T_{A_{2n-1}}\right], & n \text{ odd.}\end{array}\right\}.$$

In fact, if

$$\varphi=\varepsilon^{B_1...B_{2n-1}}_{A_1...A_{2n-1}}\left[\left[T_{B_1},...,T_{B_n}\right],T_{B_{n+1}},...,T_{B_{2n-1}}\right], \tag{62}$$

then

$$\begin{aligned}\varphi&=\varepsilon^{B_1...B_{2n-1}}_{A_1...A_{2n-1}}\left[\varepsilon^{C_1...C_n}_{B_1...B_n}T_{C_1}...T_{C_n},T_{B_{n+1}},...,T_{B_{2n-1}}\right] \\ &=\varepsilon^{B_1...B_{2n-1}}_{A_1...A_{2n-1}}\varepsilon^{C_1...C_n}_{B_1...B_n}\left[T_{C_1}...T_{C_n},T_{B_{n+1}},...,T_{B_{2n-1}}\right] \\ &=n!\varepsilon^{C_1...C_nB_{n+1}...B_{2n-1}}_{A_1..................A_{2n-1}}\left[T_{C_1}...T_{C_n},T_{B_{n+1}},...,T_{B_{2n-1}}\right]\end{aligned} \tag{63}$$

where we have used Eq.(4) and the property

$$\varepsilon^{i_1...i_r}_{h_1...h_r}B^{h_1...h_r}=r!B^{i_1...i_r}. \tag{64}$$

We consider now the multibracket $\left[T_{C_1}...T_{C_n},T_{B_{n+1}},...,T_{B_{2n-1}}\right]$. The expression $T_{C_1}...T_{C_n}$ is the matrix product of n elements, and therefore is a mapping onto another element of $\mathcal{G}$, which must be antisymmetrized together with $T_{B_{n+1}},...,T_{B_{2n-1}}$. Thus, we can write

$$\left[T_{C_1}...T_{C_n},T_{B_{n+1}},...,T_{B_{2n-1}}\right] \tag{65}$$

$$=\varepsilon^{C_{n+1}...C_{2n-1}}_{B_{n+1}...B_{2n-1}}\sum_{s=0}^{n-1}(-1)^s\,T_{C_{n+1}}...T_{C_{n+s}}T_{C_1}...T_{C_n}T_{C_{n+s+1}}...T_{C_{2n-1}}$$

where the $n-1$ elements $T_{B_{n+1}},...,T_{B_{2n-1}}$ are antisymmetrized with the contraction with $\varepsilon^{C_{n+1}...C_{2n-1}}_{B_{n+1}...B_{2n-1}}$ and the element $T_{C_1}...T_{C_n}$ is is antisymmetrized with $\sum$. Introducing these results into (63) we have

$$\begin{aligned}\varphi &= n!\varepsilon^{C_1...C_nB_{n+1}...B_{2n-1}}_{A_1.................A_{2n-1}}\varepsilon^{C_{n+1}...C_{2n-1}}_{B_{n+1}...B_{2n-1}} \\ &\times\sum_{s=0}^{n-1}(-1)^s\,T_{C_{n+1}}...T_{C_{n+s}}T_{C_1}...T_{C_n}T_{C_{n+s+1}}...T_{C_{2n-1}} \\ &= n!\,(n-1)!\varepsilon^{C_1...C_{2n-1}}_{A_1...A_{2n-1}} \\ &\times\sum_{s=0}^{n-1}(-1)^s\,T_{C_{n+1}}...T_{C_{n+s}}T_{C_1}...T_{C_n}T_{C_{n+s+1}}...T_{C_{2n-1}} \\ &= n!\,(n-1)! \\ &\times\sum_{s=0}^{n-1}(-1)^s\,\varepsilon^{C_1...C_{2n-1}}_{A_1...A_{2n-1}}T_{C_{n+1}}...T_{C_{n+s}}T_{C_1}...T_{C_n}T_{C_{n+s+1}}...T_{C_{2n-1}}\end{aligned} \tag{66}$$

where we have used the identity (64). Since

$$\begin{aligned}&\varepsilon^{C_1...C_{2n-1}}_{A_1...A_{2n-1}}T_{C_{n+1}}...T_{C_{n+s}}T_{C_1}...T_{C_n}T_{C_{n+s+1}}...T_{C_{2n-1}} \\ &= (-1)^s\,\varepsilon^{C_1...C_{2n-1}}_{A_1...A_{2n-1}}T_{C_1}T_{C_{n+1}}...T_{C_{n+s}}T_{C_2}...T_{C_n}T_{C_{n+s+1}}...T_{C_{2n-1}} \\ &= (-1)^s\,(-1)^s\,\varepsilon^{C_1...C_{2n-1}}_{A_1...A_{2n-1}}T_{C_1}T_{C_2}T_{C_{n+1}}...T_{C_{n+s}}T_{C_3}...T_{C_n}T_{C_{n+s+1}}...T_{C_{2n-1}} \\ &\vdots \\ &= (-1)^{ns}\,\varepsilon^{C_1...C_{2n-1}}_{A_1...A_{2n-1}}T_{C_1}...T_{C_n}T_{C_{n+1}}...T_{C_{n+s}}T_{C_{n+s+1}}...T_{C_{2n-1}} \\ &= (-1)^{ns}\,\varepsilon^{C_1...C_{2n-1}}_{A_1...A_{2n-1}}T_{C_1}...T_{C_{2n-1}},\end{aligned} \tag{67}$$

we have that (66) takes the form

$$\begin{aligned}\varphi &= n!\,(n-1)!\sum_{s=0}^{n-1}(-1)^s\,(-1)^{ns}\,\varepsilon^{C_1...C_{2n-1}}_{A_1...A_{2n-1}}T_{C_1}...T_{C_{2n-1}} \\ &= n!\,(n-1)!\varepsilon^{C_1...C_{2n-1}}_{A_1...A_{2n-1}}T_{C_1}...T_{C_{2n-1}}\sum_{s=0}^{n-1}(-1)^s\,(-1)^{ns} \\ &= n!\,(n-1)!\,\left[T_{A_1},...,T_{A_{2n-1}}\right]\sum_{s=0}^{n-1}(-1)^{s(n+1)}.\end{aligned}$$

It is direct to check that

$$\sum_{s=0}^{n-1}(-1)^{s(n+1)} = \left\{\begin{array}{c}0,\text{ for } n \text{ even} \\ n,\text{ for } n \text{ odd}\end{array}\right\}.$$

Using (62) we find

$$\frac{1}{n!}\frac{1}{(n-1)!}\varepsilon^{B_1 \dots B_{2n-1}}_{A_1 \dots A_{2n-1}}\left[\left[T_{B_1}, \dots, T_{B_n}\right], T_{B_{n+1}}, \dots, T_{B_{2n-1}}\right]$$
$$= \left\{ \begin{array}{ll} 0, & \text{for } n \text{ even} \\ n\left[T_{A_1}, \dots, T_{A_{2n-1}}\right], & \text{for } n \text{ odd} \end{array} \right\}$$

or

$$\frac{1}{(n-1)!}\frac{1}{n!}\sum_{\sigma\in S_{2n-1}}(-1)^{\pi(\sigma)}\left[\left[T_{A_{\sigma(1)}}, \dots, T_{A_{\sigma(n)}}\right], T_{A_{\sigma(n+1)}}, \dots, T_{A_{\sigma(2n-1)}}\right]$$
$$= \left\{ \begin{array}{ll} 0, & \text{for } n \text{ even} \\ n\left[T_{A_1}, \dots, T_{A_{2n-1}}\right], & \text{for } n \text{ odd} \end{array} \right\}.$$

[1] J.A. de Azcárraga, J.C. Pérez Bueno, Commun. Math. Phys. 184 (1997) 669 [arXiV:hep-th/9605213v3] .

[2] J.A. de Azcarraga, J.M. Izquierdo, J.C. Perez Bueno, "Talk given at 6th Fall Workshop on Geometry and Physics, Salamanca, Spain, 22-24 Sep 1997. Published in Rev.R.Acad.Cien.Exactas Fis.Nat.Ser.A Mat.95:225-248,2001.[arXiV:physics/9605213v3] .

[3] J.A. de Azcárraga and J.C. Pérez Bueno, "Talk given at 21st International Colloquium on Group Theoretical Methods in Physics, Goslar, Germany, 15-20 July, 1996.[arXiV:hep-th/9611221]

[4] F. Izaurieta, E. Rodriguez, P. Salgado, J. Math. Phys. 47 (2006) 123512 [arXiV:hep-th/0606215] .

[5] J.A. de Azcarraga, J.M. Izquierdo, M. Picon and O. Varela, Nucl. Phys. B662 (2003) 662. [arXiV:hep-th/0212347] .

[6] M. Hatsuda and M. Sakaguchi, Prog. Theor. Phys. 109, 853 (2003). [arXiV:hep-th/0106114] .

[7] J.A. de Azcarraga, J.M. Izquierdo, M. Picon and O. Varela, Int. J. Theor. Phys. 46 (2007) 2738. [arXiV:hep-th/0703017v1] .

[8] where the order of the multialgebra is denoted by n and N denotes the number of elements of the semigroup $S_E^{(N)}$.